RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2610/1

Differential Equations (Mechanics 4)

Wednesday **15 JANUARY 2003** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take $g = 9.8\ \text{m s}^{-2}$ unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

[Turn over

1 A solution is sought to the differential equation

$$\ddot{y} + 3\dot{y} + 2y = e^{kt} \sin t,$$

where k is a constant.

(i) In the case $k = 0$, find the general solution. [8]

Find also the particular solution for which $y = \dot{y} = 0$ when $t = 0$. [4]

(ii) In the case $k = 1$, verify that $y = \frac{1}{10}e^t(\sin t - \cos t)$ is a particular integral for the differential equation. Write down the general solution. [5]

(iii) Compare the behaviour of the solutions in the two cases $k = 0$ and $k = 1$ for large values of t. [2]

In the case $k = -1$, what would you expect the behaviour of the solution to be for large values of t? [1]

2 The size, w, of a rabbit population at time t years on an island with a plentiful food supply is modelled, in the absence of predators, by the differential equation

$$\frac{dw}{dt} = 2w,$$

with $w = 2000$ when $t = 0$.

(i) Solve the differential equation to find w in terms of t. Describe the behaviour of the solution and say whether this is likely to describe the actual situation. [6]

Foxes are introduced to the island. The foxes kill rabbits, but also compete with each other if the rabbit population is too small. The size of the fox population at time t years is x. The situation is now modelled by the equations

$$\frac{dw}{dt} = 2w - 80x,$$
$$\frac{dx}{dt} = 0.2x\left(1 - \frac{100x}{w}\right),$$

with $w = 2000$ and $x = 25$ when $t = 0$.

(ii) Without solving the equations, find the range of values of $\frac{w}{x}$, (i.e. the ratio of rabbits to foxes) for which

(*A*) the rabbit population increases,

(*B*) the fox population increases,

(*C*) the rabbit population increases while the fox population decreases. [5]

A numerical solution to the equations is sought using a step-by-step method. The algorithm is given by

$$t_{r+1} = t_r + h,$$
$$w_{r+1} = w_r + h\mathrm{f}(w_r, x_r),$$
$$x_{r+1} = x_r + h\mathrm{g}(w_r, x_r)$$

where $\mathrm{f}(w,x) = \frac{dw}{dt}$ and $\mathrm{g}(w,x) = \frac{dx}{dt}$.

The table shows the initial values and the results of the first iteration.

t	w	x
0	2000	25
0.1	2200	24.875
0.2		

(iii) Verify the entries for $t = 0.1$ and calculate the entries for $t = 0.2$. [6]

(iv) Explain briefly why the calculated values of x *cannot* be the actual numbers of foxes at these times. What aspect of the model has led to this inaccuracy? [3]

[Turn over

3 A parachutist of mass 80 kg falls vertically from rest from a stationary helicopter. At a distance x m below the helicopter her velocity is v m s^{-1}. The forces acting on her are her weight and air resistance of magnitude kv^2 N, where k is a constant. Her terminal velocity is 70 m s^{-1}.

(i) Show that the motion may be modelled by the differential equation

$$v\frac{dv}{dx} = 9.8 - 0.002v^2. \quad [3]$$

(ii) Solve this differential equation to show that $v = 70(1 - e^{-0.004x})^{\frac{1}{2}}$. [6]

When the parachutist's velocity reaches 99% of its terminal value, she has fallen a distance h m.

(iii) Calculate h. [1]

She then opens her parachute. The magnitude of the resistance force now changes instantly to $80v$ N.

(iv) Find her velocity in terms of t, the time in seconds since the parachute opened. Sketch a graph of v against t. [7]

(v) Calculate t when her velocity is 10 m s^{-1}. Calculate how far she falls in this time. [3]

4 A solution is sought to the differential equation

$$\frac{dy}{dt} + 2y = e^{-2t}.$$

(i) Find the complementary function. [2]

(ii) Explain why an expression of the form ae^{-2t} cannot be a particular integral of this differential equation. Find a particular integral of this differential equation. [4]

An alternative method for solving this equation is by using an integrating factor.

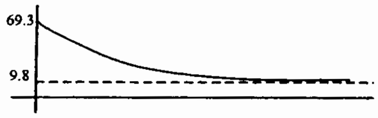
(iii) Use this method to find the general solution of the differential equation. Hence show that the particular integral found in part **(ii)** is correct. [7]

(iv) When $t = 0$, $y = y_0$. Show that the maximum value of y is $\frac{1}{2}e^{2y_0 - 1}$. State the range of values of y_0 for which this maximum occurs at a positive value of t. [7]

Mark Scheme

1(i)	$\alpha^2+3\alpha+2=0$	M1		
	$\alpha=-1$ or -2	A1		
	CF $y=Ae^{-t}+Be^{-2t}$	F1	CF for their roots (y in terms of t)	
	PI $y=a\sin t+b\cos t$	B1		
	$(-a\sin t-b\cos t)+3(a\cos t-b\sin t)+2(a\sin t+b\cos t)=\sin t$	M1	differentiate twice and substitute	
	$-a-3b+2a=1,\quad -b+3a+2b=0$	M1	compare coefficients	
	$a=\frac{1}{10}, b=-\frac{3}{10}$	A1		
	$y=Ae^{-t}+Be^{-2t}+\frac{1}{10}\sin t-\frac{3}{10}\cos t$	F1		
	$A+B-\frac{3}{10}=0$	B1	equation for A,B from their y	
	$\dot{y}=-Ae^{-t}-2Be^{-2t}+\frac{1}{10}\cos t+\frac{3}{10}\sin t$	M1	differentiate	
	$-A-2B+\frac{1}{10}=0$ and so $A=\frac{1}{2}, B=-\frac{1}{5}$	M1	substitute $t=0$ and solve	
	$y=\frac{1}{2}e^{-t}-\frac{1}{5}e^{-2t}+\frac{1}{10}\sin t-\frac{3}{10}\cos t$	A1		
				12
(ii)	$\dot{y}=\frac{1}{10}e^{t}(\sin t-\cos t)+\frac{1}{10}e^{t}(\sin t-\cos t)=\frac{1}{10}e^{t}(2\sin t)$	M1	differentiate (or use PI of correct form)	
	$\ddot{y}=\frac{1}{10}e^{t}(2\sin t+2\cos t)$	A1		
	LHS $=\frac{1}{10}e^{t}(2\cos t+2\sin t+6\sin t+2\sin t-2\cos t)$	M1	substitute in DE	
	$=e^{t}\sin t=$ RHS	E1		
	$y=Ae^{-t}+Be^{-2t}+\frac{1}{10}e^{t}(\sin t-\cos t)$	B1	general solution with their CF	
				5
(iii)	*either* $k=0\Rightarrow$ bounded oscillations	B1	For two marks, must describe (not just sketch)	
	$k=1\Rightarrow$ unbounded oscillations	B1	oscillatory behaviour and (un)boundedness.	
	or both oscillate	B1	Accept 'growing exponentially' for 'unbounded'	
	bounded for $k=0$, unbounded for $k=1$	B1	but not just 'increasing'.	
	$k=-1\Rightarrow$ solution tends to zero	B1		
				3

2(i)	solve by separating variables or CF	M1		
	$w = Ae^{2t}$	A1		
	$t = 0,\ w = 2000 \Rightarrow$	M1	use conditions	
	$w = 2000e^{2t}$	A1		
	Population grows exponentially	B1	indicate more than just 'grows'	
	either unlikely as growth will be limited by (e.g.) space/disease *or* likely as long as (e.g.) sufficient space/no disease	B1		
				6
(ii)	(A) $\dot{w} > 0 \Rightarrow 2w - 80x > 0$	M1	attempt to solve $\dot{w} > 0$	
	$\Rightarrow \frac{w}{x} > 40$	A1		
	(B) $\dot{x} > 0 \Rightarrow 1 - \frac{100x}{w} > 0$	M1	attempt to solve $\dot{x} > 0$	
	$\Rightarrow \frac{w}{x} > 100$	A1		
	(C) $40 < \frac{w}{x} < 100$	B1	correct or consistent with previous answers	
				5
(iii)	$w = 2000 + 0.1(2 \times 2000 - 80 \times 25)$	M1	demonstrate use of algorithm	
	$= 2200$	E1		
	$x = 25 + 0.1(0.2 \times 25(1 - \frac{100 \times 25}{2000}))$	M1	demonstrate use of algorithm	
	$= 24.875$	E1		
	$\dot{w} = 2410, \dot{x} = -0.6501\ldots$	M1	use algorithm again for w and x	
	$w = 2441, x = 24.81$	A1		
				6
(iv)	Actual number of foxes must be integers but values are not.	B1		
	Modelled as continuous change,	B1		
	whereas actual changes are discrete.	B1		
				3

3(i)	$mv\frac{dv}{dx} = mg - kv^2$	M1	N2L	
	$k \cdot 70^2 = mg \Rightarrow k = 0.002m\ (= 0.16)$	M1	calculate k	
	$\Rightarrow v\frac{dv}{dx} = 9.8 - 0.002v^2$	E1	clearly shown	
				3
(ii)	$\int \frac{v\,dv}{9.8 - 0.002v^2} = \int dx$	M1	separate variables	
	$-\frac{1}{0.004}\ln\lvert 9.8 - 0.002v^2 \rvert = x + c$	M1	integrate	
		A1	all correct, including constant	
	$\Rightarrow v^2 = 4900 - Ae^{-0.004x}$	M1	rearranging	
	$x = 0, v = 0 \Rightarrow A = 4900$	M1	calculate constant	
	$v = 70(1 - e^{-0.004x})^{1/2}$	E1	clearly shown	
				6
(iii)	$(1 - e^{-0.004h})^{1/2} = 0.99 \Rightarrow h \approx 979$	B1		
				1
(iv)	$m\frac{dv}{dt} = mg - 80v$	M1	N2L	
	$\int \frac{dv}{g - v} = \int dt$	M1	separate variables	
	$-\ln\lvert g - v \rvert = t + c_2$	M1	integrate	
	$\Rightarrow v = g - Be^{-t}$	A1	v in terms of t	
	$t = 0, v = 0.99 \times 70 \Rightarrow B = -59.5$	M1	calculate constant from $v(0) = 69.3$ (or 70)	
	$v = 9.8 + 59.5e^{-t}$	A1	cao	
		B1	intercept and asymptote labelled	
				7
(v)	$v = 10 \Rightarrow t = 5.70\ldots$	B1		
	$x = \int_0^{5.70} (9.8 + 59.5e^{-t})\,dt$	M1	integrate v between limits	
	$\approx 115\text{ m}$	A1	cao	
				3

4(i)	$\alpha + 2 = 0 \Rightarrow \alpha = -2$	M1	solve auxiliary equation	
	CF $y = A\,e^{-2t}$	F1		
				2
(ii)	It is the same form as the CF so satisfies homogeneous equation, hence will not satisfy the non-homogenous equation.	B1	justifies that it will not satisfy DE (may use substitution)	
	$y = at\,e^{-2t}$	B1	correct PI	
	in DE: $a\,e^{-2t} - 2at\,e^{-2t} + 2at\,e^{-2t} = e^{-2t} \Rightarrow a = 1$	M1	differentiate, substitute and compare coefficients	
	PI $y = t\,e^{-2t}$	A1		
				4
(iii)	$I = \exp(\int 2\,dt)$	M1	attempt integrating factor	
	$= e^{2t}$	A1		
	$e^{2t}\frac{dy}{dt} + 2e^{2t}y = 1$	M1	multiply	
	$y\,e^{2t} = \int dt$	M1	integrate	
	$y\,e^{2t} = t + A$	A1		
	$y = t\,e^{-2t} + A\,e^{-2t}$	A1		
	i.e. CF $A\,e^{-2t}$, PI $t\,e^{-2t}$ as before	E1	correctly identify PI	
				7
(iv)	condition $\Rightarrow A = y_0$	M1	calculate constant	
	$y = (t + y_0)e^{-2t}$	F1	particular solution	
	$0 = \frac{dy}{dt} = (1 - 2t - 2y_0)e^{-2t}$	M1	set derivative to zero	
	$\Rightarrow t = \frac{1}{2} - y_0$	M1	solve for t	
	$t < \frac{1}{2} - y_0 \Rightarrow \frac{dy}{dt} > 0,\ t > \frac{1}{2} - y_0 \Rightarrow \frac{dy}{dt} < 0$ hence maximum	E1	or alternative justification	
	$y_{max} = (\frac{1}{2} - y_0 + y_0)e^{-2(\frac{1}{2} - y_0)} = \frac{1}{2}e^{2y_0 - 1}$	E1	clearly shown	
	$\frac{1}{2} - y_0 > 0 \Rightarrow y_0 < \frac{1}{2}$	B1		
				7

Examiner’s Report

2610 Mechanics 4

General Comments

The standard of scripts was generally high, with very few weak scripts. The mathematics was often accurate and logically presented. Question 1 was the most popular with virtually all candidates attempting it.

Comments on Individual Questions

Question 1

This question attracted many excellent solutions. Most candidates were able to solve the equations, although minor slips were common. Candidates should be reminded that an instruction to 'verify' does not require them to derive the given result. It was quite sufficient to substitute the particular integral into the differential equation and show that it produced the correct result. When commenting on the behaviour of the solutions, candidates were not expected to describe the solutions in great detail but they were expected to observe that both oscillate and that one is bounded and the other is unbounded.

(i) $y = \frac{1}{2}e^{-t} - \frac{1}{5}e^{-2t} + \frac{1}{10}\sin t - \frac{3}{10}\cos t$; (ii) $y = Ae^{-t} + Be^{-2t} + \frac{1}{10}e^{t}(\sin t - \cos t)$.

Question 2

Candidates usually were able to solve the differential equation and comment about the behaviour it predicts. Some candidates did not realise that the signs of $\dot{x}$ and $\dot{w}$ needed to be considered in establishing the required ratios of rabbits to foxes. Those candidates who did, usually were able to deal with the first case, but sometimes confused the signs when dealing with $\dot{x}$. Candidates were usually able to carry out the numerical algorithm, but not always entirely accurately. The verified results sometimes were lacking in sufficient working to be convincing. In the final part of the question, only a minority of candidates identified that the number of foxes had to be an integer. Instead of discussing the accuracy of the model (as the question asked), many candidates discussed the accuracy of the algorithm.

(i) $w = 2000e^{2t}$; (ii)(*A*) $\frac{w}{x} > 40$, (*B*) $\frac{w}{x} > 100$, (*C*) $40 < \frac{w}{x} < 100$; (iii) $w = 2441, x = 24.81$.

Question 3

Candidates usually produced the desired differential equation, but sometimes by stating the value of k without the necessary justification. When solving the equation, some candidates omitted the arbitrary constant but still managed to produce the given solution! Some candidates struggled with the integration. However there were many good solutions. Solving for velocity in terms of time was also often done well, although again some candidates omitted the constant. Sketches of the solution were sometimes weak. Candidates were expected to indicate the horizontal asymptote, and the initial value for velocity should be labelled and be consistent with the initial conditions for the problem.

(iii) 979; (iv) $v = 9.8 + 59.5e^{-t}$; (v) 115 m.

Question 4

The first part of the question presented few problems. Explanations as to why the suggested expression could not be a particular integral were sometimes vague and incomplete. Candidates generally knew the correct form of the particular integral but were not always able to complete the solution accurately. Most candidates could successfully use the integrating factor method, but were not always able to relate the solution to the particular integral found previously. In the final part of the question, candidates sometimes did not deal with the arbitrary constant, and generally candidates found difficulties in maintaining accuracy in their solution.

(i) $y = Ae^{-2t}$; (ii) $y = te^{-2t}$; (iii) $y = te^{-2t} + Ae^{-2t}$; (iv) $y_0 < \frac{1}{2}$.